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HEAT TRANSFER TO LAMINAR FLOW IN A ROUND TUBE OR FLAT CONDUIT - - - THE GRAETZ PROBLEM EXTENDED

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HEAT TRANSFER TO LAMINAR FLOW IN A ROUND TUBE OR FLAT CONDUIT --- THE GRAETZ PROBLEM EXTENDED

By. hn Sellars, Myron Tribus, and John Klein

NOMENCLATURE

The following nomenclature is used in the paper:

A = Coefficient occurring in Equation (10).

B = Coefficient occurring in Equation (10).

b = Half width of flat duct, ft.

C_n = Coefficient in Equation (3).

C_p = Unit heat capacity at constant pressure, Btu/lb-°F.

D = Coefficient occurring in Equation (18).

E = Coefficient occurring in Equation (18).

F = $F(s, r^+)$, Laplace transform of Graetz solution.

g = $g(x^+, r^+)$ Integrating kernel for heat-flux problems, see Equation (44).

G = $G(s, r^+)$ Laplace transform of g.

h = $h(x^+)$ Integrating kernel for heat-flux problems, see Equation (34).

H = $H(s)$ Laplace transform of h.

i = $\sqrt{-1}$.

J = Bessel function of first kind, zero order.

$J_{1/3}$ = Bessel function of first kind, 1/3 order.

$J_{-1/3}$ = Bessel function of first kind, -1/3 order.

k = Thermal conductivity of fluid, Btu/sec-ft² (°F/ft).

K_n = Coefficient occurring in Equation (A-3).

Pr = Prandtl modulus, dimensionless, $(\mu C_p/k)(3600\ g_c)$.

q = $q(x)$ heat flux per unit wall area, Btu/hr-ft².

Q = Laplace transform of (kq/r_o).

r = Radius, ft.

r_o = Tube radius, ft.

r^+ = (r/r_o).

R = $R(r^+)$.

R_e = Reynolds modulus, dimensionless, $(2U_m r_o\rho/\mu g_c)$ or $(4U_m b\rho/\mu g_c)$.

s = Transform variable.

t = $t(x^+, r^+)$ temperature, °F.

T = $T(s, r^+)$, Laplace transform of t.

u = Velocity of fluid, ft/sec.

u_m = Average fluid velocity in tube, ft/sec.

x = *Distance along tube, ft.*

x^+ = $(x/r_o)(RePr)^{-1}$ or $(x/b)(RePr)^{-1}$.

y = Distance from duct wall, ft.

y^+ = (y/b).

z = Distance from tube wall, ft.

z^+ = (z/r_o).

γ = Zero of $H(s)$.

λ = Eigenvalue.

μ = Viscosity of fluid, lb-sec/ft².

ρ = Fluid density, lb/ft³.

ξ = Dummy variable.

η = Dummy Variable.

Γ = Gamma function.

INTRODUCTION

The problem considered here is posed by a system in which a fluid of constant properties flows in steady laminar motion in a round tube or flat duct. The velocity profile is fully established and parabolic. Up to a point ($x = 0$) the fluid is isothermal. After this point a prescribed heat flux or temperature is given at the wall of the conduit and the problem is to find the temperature distribution, as well as the connection between heat flux and wall temperature. The application of this solution to practical problems of heat exchange has already been so well established that further comment is unnecessary.

The problem has been considered in detail by a number of workers and an excellent review is contained in the book "Heat Transfer" by M. Jakob.[1] The problem readily reduces to the finding of eigenvalues and prior to this paper only the first three eigenfunctions and the first four eigenvalues have been known. A recent paper[2] has brought out the importance of obtaining more eigenvalues, and by using the complete set of eigenvalues and the methods of reference 2 the classical "Graetz Problem"[3] is extended to more complicated boundary conditions.

The problem can be stated in mathematical terms as follows:

Given

$$t = t(x, r)$$

$$u \rho c_p \frac{\partial t}{\partial x} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial t}{\partial r}\right) \tag{1}$$

$$u = 2 u_m \left[1 - (r/r_0)^2\right]$$

and for

$$t = t_0 \qquad x < 0$$

with either

$$\left.\begin{aligned} t(x, r_0) &= t_w(x) \\ \text{or} \quad k\, t_r(x, r_0) &= q(x) \end{aligned}\right\} \quad x > 0$$

find t(x,r) and the relation between q(x) and $t_w(x)$.

The nondimensional form of the equations is

$$\frac{\partial t}{\partial x^+} = \frac{1}{1 - r^{+2}} \frac{1}{r^+} \frac{\partial}{\partial r^+}\left(r^+ \frac{\partial t}{\partial r^+}\right) \tag{2}$$

The boundary conditions are

$$t = t_0 \qquad x^+ < 0$$

and either

$$\left.\begin{aligned} t(x^+, 1) &= t_w(x^+) \\ \text{or} \quad t_{r^+}(x^+, 1) &= r_0\, q(x^+)/k \end{aligned}\right\} \quad x^+ > 0$$

In view of the linearity of Equation (2), it is necessary to have only the fundamental solution, known as the Graetz solution, to construct all other needed solutions. Therefore, the initial step is the completion of the Graetz solution.

1 'Heat Transfer,' by Max Jakob, John Wiley & Sons, Inc., New York, N. Y., vol. 1, 1949.

THE GRAETZ SOLUTION

The problem considered by Graetz and most other workers is Equation (2) with boundary conditions.

$$t = 1 \qquad x^+ < 0$$

$$t_w = 0 \qquad x^+ > 0$$

Led Θ be a solution of Equation (2), then

$$\Theta = \sum_{n=0}^{\infty} C_n R_n(r^+) e^{-\lambda_n^2 x^+} \tag{3}$$

where the λ_n are the eigenvalues required to make the solution to the following differential equation

$$r^+ R_n'' + R_n' + \lambda_n^2 r^+ (1 - r^{+2}) R_n = 0 \tag{4}$$

satisfying the boundary conditions $R_n(1) = 0$, $R_n(0) = 1$. The coefficients C_n are determined from the relation

$$C_n = \frac{\int_0^1 r^+ (1 - r^{+2}) R_n \, dr^+}{\int_0^1 r^+ (1 - r^{+2}) R_n^2 \, dr^+} = \frac{-2}{\lambda_n \left(\frac{\partial R_n}{\partial \lambda} \right)_{\substack{r^+ = 1 \\ \lambda = \lambda_n}}} \tag{5}$$

The eigenfunctions and eigenvalues have been given only for n = 1, 2, 3. The higher modes of Equation (4) are very difficult to calculate for large values of λ . Therefore, to obtain λ_n and C_n for $n > 3$, a solution is sought which will be valid as $\lambda_n \to \infty$. It will be found that the resulting formulae will provide good answers even when λ_n is small. First, look for a solution in the form

$$R = e^{g(r^+)}$$

and find that $g(r^+)$ satisfies

$$g'' + g'^2 + \frac{1}{r^+} g' + \lambda^2 (1 - r^{+2}) = 0 \tag{6}$$

Now an asymptotic solution is sought in the form

$$g = \lambda g_0 + g_1 + \lambda^{-1} g_2 + \cdots \tag{7}$$

Substitution in Equation (6) and equating powers of λ gives

$$g_0' = \pm i \sqrt{1 - r^{+2}} \tag{8}$$

$$g_1 = -\ln \sqrt{g_0' r^+} \tag{9}$$

Since λ is large, the remaining terms in Equation (7) are neglected. Substitution of Equations (8) and (9) in Equation (7) gives for R

$$R = \frac{A e^{i\lambda \int_0^{r^+} \sqrt{1-\xi^2}\, d\xi} + B e^{-i\lambda \int_0^{r^+} \sqrt{1-\xi^2}\, d\xi}}{\sqrt{r^+}\,(1-r^{+2})^{1/4}} \tag{10}$$

Equation (10) is the so-called WKB approximation and is valid for $o < r^+ < 1$ for sufficiently large λ. Now the coefficients A and B must be determined so that Equation (10) will correspond to the regular solution of Equation (4), where r^+ is small. For small r^+ Equation (10) is

$$R = \frac{A e^{i\lambda r^+} + B e^{-i\lambda r^+}}{\sqrt{r^+}\,(1-r^{+2})^{1/4}} \tag{11}$$

Inspection of Equation (4) shows that when r^+ is small enough so that $\lambda^2 (1 - r^{+2}) \to \lambda^2$, the classical solution behaves as $J_o (\lambda r^+)$, since Equation (4) then becomes a Bessel equation. For large λr^+, even if r^+ is small, the asymptotic expression for $J_o(\lambda r^+)$ is

$$J_0(\lambda r^+) = \sqrt{\frac{2}{\pi \lambda r^+}}\, \cos(\lambda r^+ - \pi/4) \tag{12}$$

and thus, it is seen that to make Equations (11) and (12) equal for r^+ small, it is required that

$$A = \sqrt{\frac{2}{\lambda\pi}}\, e^{-i\pi/4} \qquad B = \sqrt{\frac{2}{\lambda\pi}}\, e^{i\pi/4} \tag{13}$$

and for $0 < r^+ < 1$

$$R(r^+) = \sqrt{\frac{2}{\pi\lambda r^+}}\ \frac{\cos\left(\lambda \int_0^{r^+} \sqrt{1-\xi^2}\, d\xi - \pi/4\right)}{(1-r^{+2})^{1/4}} \tag{14}$$

Equation (14) is not a good approximation to the solution as $r^+ \to 1$, since it has a singularity there. Because a boundary condition is to be imposed at $r^+ = 1$, the development of an alternate solution, valid near $r^+ = 1$, is considered. By patching it on to Equation (14) the solution over the range $0 \leq r^+ \leq 1$ is obtained.

The following change of variable is made

$$z^+ = 1 - r^+$$

and Equation (4) becomes

$$\frac{d^2R}{dz^{+2}} - \frac{1}{1-z^+}\frac{dR}{dz^+} + \lambda^2 z^+(2-z^+)R = 0 \qquad (15)$$

Now consider $0 < z^+ \ll 1$ and define a new variable

$$\eta = \lambda^{2/3} z^+ \qquad (16)$$

Substitution of Equation (16) into Equation (14) yields for large λ

$$\frac{d^2R}{d\eta^2} + 2\eta R = 0 \qquad (17)$$

which has the solution

$$R = D\sqrt{z^+}\, J_{1/3}\left(\frac{\lambda\sqrt{8}}{3} z^{+3/2}\right) + E\sqrt{z^+}\, J_{-1/3}\left(\frac{\lambda\sqrt{8}}{3} z^{+3/2}\right) \qquad (18)$$

The constants D and E are to be so chosen that for small z^+ Equations (18) and (14) are equivalent.

Change the variable from r^+ to z^+ in Equation (14) and perform the integration

$$\int_0^{r^+}\sqrt{1-\xi^2}\,d\xi = \int_0^1\sqrt{1-\xi^2}\,d\xi + \int_1^{r^+}\sqrt{1-\xi^2}\,d\xi = \pi/4 - \int_0^{z^+}\sqrt{2\xi-\xi^2}\,d\xi \qquad (19)$$

For small z^+ Equation (19) yields

$$\int_0^{r^+}\sqrt{1-\xi^2}\,d\xi = \pi/4 - \sqrt{8}/3\; z^{+3/2} \qquad (20)$$

so that Equation (14) for small z^+ is

$$R(z^+) = \sqrt{\frac{2}{\pi\lambda}}\;\frac{\cos\left(\sqrt{8}/3\,\lambda z^{+3/2} - (\lambda-1)\pi/4\right)}{2^{1/4} z^{+1/4}} \qquad (21)$$

For large λz^+, even if z^+ is small, Equation (18) becomes

$$R(z^+) = \sqrt{\frac{3}{\pi\lambda}}\;\frac{D\cos\left(\frac{\lambda\sqrt{8}}{3} z^{+3/2} - 5\pi/12\right) + E\cos\left(\frac{\lambda\sqrt{8}}{3} z^{+3/2} - \pi/12\right)}{2^{1/4} z^{+1/4}} \qquad (22)$$

Expanding the cosines of differences of angle occurring in Equations (21) and (22) yields the simultaneous equation

$$D\cos\frac{5\pi}{12} + E\cos\pi/12 = \sqrt{2/3}\,\cos(\lambda-1)\pi/4$$

$$D \sin \frac{5\pi}{12} + E \sin \pi/12 = \sqrt{3/3} \sin (\lambda-1)\pi/4 \tag{23}$$

from which D and E are evaluated. Therefore, Equation (18) is

$$R(z^+) = 2/3 \sqrt{2z^+} \left[\sin\left(\frac{\lambda\pi}{4} - \frac{\pi}{3}\right) J_{1/3}\left(\frac{\lambda\sqrt{8}}{3} z^{+3/2}\right) - \sin\left(\frac{\lambda\pi}{4} - \frac{2\pi}{3}\right) J_{-1/3}\left(\frac{\lambda\sqrt{8}}{3} z^{+3/2}\right) \right. \tag{24}$$

As $z^+ \to 0$ the product $\sqrt{z^+}\, J_{1/3} [\lambda \sqrt{8}/3\ z^{+3/2}] \to 0$, but the product involving $J_{-1/3}$ becomes constant. Therefore, the coefficient of $J_{-1/3}$ must be zero if $R = 0$ at $z^+ = 0$. The values of λ_n must therefore be given by

$$\lambda_n = 4n + 8/3 \qquad n = 0, 1, 2, \cdots \tag{25}$$

The equations for R_n are therefore

for small r^+ (center of pipe)

$$R_n(r^+) = J_0(\lambda_n r^+) \tag{26}$$

for medium r^+

$$R_n(r^+) = \sqrt{\frac{2}{\pi\lambda_n r^+}}\ \frac{\cos \frac{\lambda_n}{2} r^+\sqrt{1-r^{+2}} + \lambda_n/2 \arcsin r^+ - \pi/4}{(1-r^{+2})^{1/4}} \tag{27}$$

and for small $z^+ = 1 - r^+$ (near the wall)

$$R_n(z^+) = \sqrt{\frac{2z^+}{3}}\ (-1)^n\ J_{1/3}\left(\frac{\lambda_n\sqrt{8}}{3} z^{+3/2}\right) \tag{28}$$

Equations (24) to (28) contain all the information essential to the problem solution. The coefficients C_n in Equation (3) are found from Equation (24) in accordance with Equation (5). Thus it is found that

$$\left(\frac{\partial R}{\partial \lambda}\right)_{\substack{\lambda=\lambda_n \\ z^+=0}} = (-1)^{n+1} \frac{\pi\lambda_n^{-1/3}}{6^{2/3}\,\Gamma(2/3)} \qquad n = 0, 1, 2, \cdots \tag{29}$$

and therefore

$$C_n = (-1)^n \frac{2 \cdot 6^{2/3}\, \Gamma(2/3)}{\pi} \lambda_n^{-2/3} \qquad n = 0, 1, 2, \cdots \tag{30}$$

The derivative of R at the wall ($z^+ = 0$) which is

$$R_n'(1) = -\left(\frac{\partial R_n}{\partial z^+}\right)_{z^+=0} = \frac{(-1)^{n+1}\, 2^{2/3}\, \lambda_n^{1/3}}{\Gamma(1/3)\, 3^{5/6}} \qquad n = 0, 1, 2, \cdots \tag{31}$$

will be required later.

Table I shows the first ten eigenvalues and the important constants for the case of flow in a round tube. Table II gives the same data for a flat duct with opposite walls at the same temperature. The development of the flat-duct system is similar to the round duct and the equations are given in the appendix, numbered to correspond with the text.

TABLE I FIRST TEN EIGENVALUES AND THE IMPORTANT CONSTANTS FOR THE CASE OF FLOW IN A ROUND TUBE

n	λ_n	λ_n^2	C_n	$-1/2\ C_n R_n'(1)$
0	2 2/3	7.1129	+1.47989	0.7303
1	6 2/3	44.489	-0.80345	0.53810
2	10 2/3	113.785	+0.58732	0.460074
3	14 2/3	215.121	-0.474993	0.413743
4	18 2/3	348.457	+0.404448	0.381785
5	22 2/3	513.793	-0.355345	0.357853
6	26 2/3	711.129	+0.318858	0.338988
7	30 2/3	940.465	-0.290488	0.323555
8	34 2/3	1201.8	+0.267691	0.310596
9	38 2/3	1495.1	-0.248895	0.29950

$$C_m = (-1)^m \frac{2 \cdot 6^{2/3}\, \Gamma(2/3)\, \lambda_m^{-2/3}}{\pi} = (-1)^m\, 2.84606\, \lambda_m^{-2/3}$$

$$-\frac{C_m}{2} R_m'(1) = \frac{6^{2/3}\, \Gamma(2/3)\, 2^{2/3}\, \lambda_m^{-1/3}}{\pi\, \Gamma(4/3)\, 3^{5/6}} = 1.01276\, \lambda_m^{-1/3}$$

$$\lambda_m = 4m + 8/3 \qquad m = 0, 1, 2, \ldots$$

$$\theta = \Sigma\, C_m R_m(r^+)\, e^{-\lambda_m^2 x^+}$$

$$q(x^+) = -\frac{4k}{d} \Sigma \frac{C_m}{2} R_m'(1)\, e^{-\lambda_m^2 x^+} (t_w - t_o)$$

The previously known eigenvalues given by Jakob are shown in Table III for comparison. Since the solution presented here is valid for large λ_n,

TABLE II FIRST TEN EIGENVALUES AND THE IMPORTANT CONSTANTS FOR THE CASE OF FLOW IN A FLAT DUCT WITH OPPOSITE WALLS AT THE SAME TEMPERATURE

n	λ_n	λ_n^2	K_n	$-K_n Y_n'(1)$
0	1.667	2.779	+0.503	.683
1	5.667	32.11	-0.121	.454
2	9.667	93.45	+0.0648	.380
3	13.67	186.9	-0.0431	.338
4	17.67	312.2	+0.0319	.311
5	21.67	469.6	-0.0253	.291
6	25.67	658.9	+0.0207	.274
7	29.67	880.3	-0.0174	.262
8	33.67	1134	+0.0150	.251
9	37.67	1419	-0.0131	.242

$$K_m = (-1)^m \frac{3^{2/3}\,\Gamma(2/3)\,2^{13/6}}{\pi^{3/2}} \lambda_m^{-7/6} = (-1)^m\, 0.913\, \lambda_m^{-7/6}$$

$$-K_m Y_m'(1) = \frac{4 \cdot 2^{1/3}\,\Gamma(2/3)\,\lambda_m^{-1/3}}{\pi\,\Gamma(4/3)\,3^{1/6}} = 0.810\,\lambda_m^{-1/3}$$

$$\lambda_m = 4m + 5/3 \qquad m = 0, 1, 2, \cdots \qquad \theta = \Sigma K_m Y_m(y^+)\, e^{-\lambda_m^2 \frac{8}{3} x^+}$$

$$q(x^+) = \Sigma -\frac{k}{b} K_m Y_m'(1)\, e^{-\frac{8}{3}\lambda_m^2 x^+} (t_w - t_o)$$

TABLE III COMPARISON WITH PREVIOUSLY KNOWN EIGENVALUES

	Results Obtained								
	Sellars, Tribus, Klein			Jakob			Analogue Computer		
n	λ_n	C_n	$\frac{-C_n R_n'(1)}{2}$	λ_n	C_n	$\frac{-C_n R_n'(1)}{2}$	λ_n	C_n	$\frac{-C_n R_n'(1)}{2}$
0	2.667	+1.47989	0.7303	2.705	+1.477	0.749	2.71	1.46	0.735
1	6.667	-0.80345	0.5381	6.66	-0.810	0.539	6.69	-0.809	0.533
2	10.667	+0.58732	0.4601	10.3	+0.385	0.179	10.62	+0.592	0.444
3	14.667	-0.47499	0.4137	14.67*	-0.479*	-----	14.58	-0.51	0.398

*Attributed to Lee, Nelson, Cherry and Boelter.

and in view of the agreement even at moderate values of λ_n, it has been concluded that all the eigenvalues and functions are now sufficiently accurately known.

The heat flux at the wall is computed from the equation

$$q(x^+) = k\left(\frac{\partial t}{\partial r^+}\right)_{r^+=1} = \frac{-4k}{d} \sum \frac{c_n}{2} R_n'(1)\, e^{-\lambda_n x^+} (t_w - t_o) \tag{32}$$

Equation (32) is presented in the above form to bring it into agreement with Jakob.[1]

ARBITRARY WALL-TEMPERATURE VARIATIONS

If the wall-temperature variation is given by $t_w(x)$, then, as shown by Tribus and Klein,[2] the principle of superposition may be applied and the solution may be written in a Fourier-type Stieltjes integral

$$t - t_o = \int_{\xi=0}^{x^+} [1 - \theta(x^+ - \xi, r^+)]\, dt_w(\xi) \tag{33}$$

where θ is the solution to Equation (2) defined by Equation (3). The temperature of the wall and fluid for $x^+ < 0$ is t_o. The Stieljes integral in Equation (33) is evaluated by substituting $(dt_w/d\xi)\, d\xi$ for dt_w wherever t_w is continuous and substituting $[1 - \theta\, (x^+ - \xi_1, r^+)][t\,(\xi_1^+) - t\,(\xi_1^-)]$ as the contribution of the integral wherever $t_w\,(x^+)$ has a discontinuity. (See Tribus and Klein[2] for a more detailed discussion.) The heat flux is computed from

$$q(x^+) = k\left(\frac{\partial t}{\partial r^+}\right)_{r^+=1} = \frac{-k}{r_o} \int_0^{x^+} \theta_r (x^+ - \xi, 1)\, dt_w(\xi) \tag{34}$$

HEAT FLUX AT THE WALL GIVEN

The inverse problem; namely, "Given the heat flux at the wall, what is the temperature?", may be solved with the aid of the Laplace transform theory Define the following transforms

$$T(s, r^+) = \int_0^\infty e^{-sx^+} (t - t_o)\, dx^+ \tag{35}$$

$$T_w(s) = T(s, 1) \tag{36}$$

[2] Forced Convection from Nonisothermal Surfaces, by M. Tribus and J. Klein, Heat Transfer: A Symposium held at the University of Michigan during the Summer of 1952, Engineering Research Institute, University of Michigan, 1953, pp 211-235.

$$F(s, r^+) = \int_0^\infty [1 - \theta(x^+, r^+)] e^{-sx^+} dx^+ \tag{37}$$

$$H(s) = T_{r^+}(s, 1) = -\int_0^\infty \theta_{r^+}(x^+, 1) e^{-sx^+} dx^+ \tag{38}$$

$$Q(s) = \frac{r_o}{k} \int_0^\infty e^{-sx^+} q(x^+) dx^+ \tag{39}$$

Applying the Faltung theorem to Equations (33) and (34) yields;

$$T(s, r^+) = F(s, r^+) \, s \, T_w(s) \tag{40}$$

$$Q(s) = H(s) \, s \, T_w(s) \tag{41}$$

If the heat flux is finite, $t_w(x^+)$ will be continuous. Eliminating $t_w(s)$ from the above equations,

$$T(s, r^+) = \frac{F(s, r^+)}{H(s)} Q(s) \tag{42}$$

Now define

$$G(s, r^+) = \frac{F(s, r^+)}{H(s)} \tag{43}$$

and let $g(x^+, r^+)$ be the inverse transform of $G(s, r^+)$. Then, for arbitrary heat flux at the wall, the temperature is given by

$$t - t_o = \frac{r_o}{k} \int_0^{x^+} g(x^+ - \xi, r^+) \, q(\xi) \, d\xi \tag{44}$$

Thus, the problem is reduced to finding $g(x^+, r^+)$, which is given by

$$g(x^+, r^+) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx^+} \frac{F(s, r^+)}{H(s)} \, ds \tag{45}$$

Returning to Equations (37), (38), and (3) it is found that

$$F(s, r^+) = \frac{1}{s} - \sum_{n=0}^{\infty} \frac{C_n R_n(r^+)}{s + \lambda_n^2} \tag{46}$$

$$H(s) = -\sum_{n=0}^{\infty} \frac{C_n R_n'(1)}{s + \lambda_n^2} \tag{47}$$

Because F and H have poles at $s = -\lambda^2_n$, the quotient F/H has no poles except at S = 0 and the zeroes of H(s) and the zeroes of H(s) must be found nu-

merically. Because H'(s) is monotonic, it is found that the zeroes of H(s) occur between the $-\lambda^2_n$. Letting γ^2_m be the values satisfying $H(-\gamma^2_m) = 0$, from the theory of residues

$$g(x, \lambda^+) = \frac{1}{H(0)} - \sum_m \frac{e^{-\gamma_m^2 x^+}}{\gamma_m^2 H'(-\gamma_m^2)} - \sum_m c_m R_m(\lambda^+) \sum_m \frac{e^{-\gamma_m^2 x^+}}{\lambda_m^2 - \gamma_m^2} \tag{48}$$

Table IV gives the values of γ^2_m, $H'(-\gamma^2_m)$ for the first three values of m. The term H(0) has been shown by others[1] to be given by

$$H(0) = +\ 1/4 \tag{49}$$

hence, the wall temperature may be easily calculated with the aid of

$$g(x^+, 1) = 4 - \sum_m \frac{e^{-\gamma_m^2 x^+}}{\gamma_m^2 H'(-\gamma_m^2)} \tag{50}$$

TABLE IV THE VALUES OF γ^2_m, $H'(-\gamma^2_m)$ FOR THE FIRST THREE VALUES OF m

Roots of H(s) = 0, Values of $H'(-\gamma^2_m)$

$$H(s) = -\sum_{m=0}^{\infty} \frac{c_m R_m'(1)}{s + \lambda_m^2}$$

$$c_m R_m'(1) = -2.02552\ \lambda_m^{-1/3}$$

$$\lambda_m = 4m + 8/3 \qquad m = 0, 1, 2, \cdots$$

$$H'(s) = \sum_m \frac{c_m R_m'(1)}{(s + \lambda_m^2)^2}$$

m	γ^2_m	$-H'(-\gamma^2_m)$	$\frac{-1}{\gamma^2_m H'(-\gamma^2_m)}$
1	25.639	8.854×10^{-3}	4.405
2	84.624	2.062×10^{-3}	5.7308
3	176.40	9.435×10^{-4}	6.0084

TABLE V VALUES OF V^2_m, $H'(-\gamma^2_m)$ FOR THE FIRST THREE VALUES OF m FOR A FLAT DUCT

Roots of $\bar{H}(s) = 0$, Values of $\bar{H}'(-\gamma^2_m)$

$$\bar{H}(s) = -\sum \frac{K_m Y_m'(1)}{s + \frac{8}{3}\lambda_m^2}$$

$$\bar{H}'(s) = \sum \frac{K_m Y_m'(1)}{(s + \frac{8}{3}\lambda_m^2)^2}$$

$$\lambda_m = 4m + 5/3 \qquad m = 0, 1, 2, \cdots$$

m	γ^2_m	$-\bar{H}'(-\gamma^2_m)$	$\frac{-1}{\gamma^2_m \bar{H}'(-\gamma^2_m)}$
1	49.345	7.45×10^{-4}	27.2
2	185.94	1.67×10^{-4}	32.1
3	409.45	6.89×10^{-5}	35.4

A SAMPLE CALCULATION FOR CONSTANT WALL HEAT FLUX

By way of illustration consider the computation of the asymptotic value of the Nusselt modulus for the case of constant heat flux at the wall. Combining Equations (44) and (48) with $q(\xi) = q$ = constant, the following is obtained.

$$t(x^+, r^+) - t_0 = \frac{q r_0}{k} \int_0^{x^+} \left[4 - \sum_m \frac{e^{-\gamma_m^2 (x^+ - \xi^+)}}{\gamma_m^2 H'(-\gamma_m^2)} - \sum_n c_n R_n \sum_m \frac{e^{-\gamma_m^2 (x^+ - \xi^+)}}{\lambda_n^2 - \gamma_m^2} \right] d\xi^+ \tag{51}$$

Letting $\beta x = x^+$, where $\beta = \pi k / 2 W C_p$, and integrating Equation (51) gives

$$t(x^+, r^+) - t_0 = \frac{q r_0}{k} \left\{ 4\beta x - \sum_m \frac{1 - e^{-\gamma_m^2 \beta x}}{\gamma_m^4 \, H'(-\gamma_m^2)} - \sum_n c_n R_n \sum_m \frac{1 - e^{-\gamma_m^2 \beta x}}{\gamma_m^2 (\lambda_n^2 - \gamma_m^2)} \right. \tag{52}$$

which may be rewritten as

$$t(x^+, r^+) - t_0 = \frac{q r_0}{k} \left[4\beta x - \sum_m \frac{1}{\gamma_m^4 \, H'(-\gamma_m^2)} + \sum_m \frac{e^{-\gamma_m^2 \beta x}}{\gamma_m^4 \, H'(-\gamma_m^2)} - \sum_n c_n R_n \sum_m \frac{1 - e^{-\gamma_m^2 \beta x}}{\gamma_m^2 (\lambda_n^2 - \gamma_m^2)} \right] \tag{53}$$

Equation (53) shows that far down the pipe ($x^+ \to \infty$) the derivative of t with respect to x is independent of x or r^+; i.e.,

$$\frac{\partial t}{\partial x} = \frac{4\beta q r_0}{k} \qquad \text{for} \quad x^+ \longrightarrow \infty \tag{54}$$

Substituting this quantity into Equation (2) leads to

$$\frac{4 q r_0}{k} = \frac{1}{r^+ (1 - r^{+2})} \frac{\partial}{\partial r^+} \left(r^+ \frac{\partial t}{\partial r^+} \right) \tag{55}$$

which may be integrated directly to give

$$t(x^+, r^+) - t(x^+, 0) = \frac{4 r_0 q}{k} \left(\frac{r^{+2}}{4} - \frac{r^+}{16} \right) \tag{56}$$

Now the mixed mean temperature along the pipe is given by

$$t_{mm}(x^+) - t_0 = \frac{2\pi r_0 q x}{W C_p} = \frac{4 r_0 q}{k} \beta x \tag{57}$$

but the mixed mean temperature is also defined by

$$t_{mm}(x^+) = \frac{\int_0^1 u \rho c_p \, t(x^+, r^+) \, 2\pi r^+ dr^+}{W c_p} \tag{58}$$

Substituting Equation (56) into Equation (58) and integrating results in

$$t_{mm}(x^+) - t(x^+, 0) = \frac{7}{24} \frac{r_0 q}{k} \tag{59}$$

Combining Equations (59) and (56)

$$t(x^+, r^+) = \frac{4 r_0 q}{k} \left(\frac{r^{+2}}{4} - \frac{r^{+4}}{16} \right) + t_{mm} - \frac{7}{24} \frac{r_0 q}{k} \tag{60}$$

and substituting Equation (57) into Equation (60)

$$t(x^+, r^+) - t_0 = \frac{r_0 q}{k} \left[4\beta x + r^{+2} - \frac{r^{+4}}{4} - \frac{7}{24} \right] \tag{61}$$

at $r^+ = 1$. This expression reduces to

$$t(x^+, 1) - t_0 = \frac{r_0 q}{k} \left[4\beta x + \frac{11}{24} \right] \tag{62}$$

but from Equation (52), since $R_n(1) = 0$,

$$t(x^+, 1) - t_0 = \frac{r_0 q}{k} \left[4\beta x - \sum_m \frac{1}{\gamma_m^4 \, H'(-\gamma_m^2)} \right] \tag{63}$$

Hence,

$$\sum \frac{1}{\gamma_m^4 \, H'(-\gamma_m^2)} = -\frac{11}{24} \cong -0.458 \tag{64}$$

(Note that the first three terms sum to approximately -0.27.) Now, the Nusselt modulus is given by

$$Nu = \frac{2 r_0 q}{k(t_w - t_{mm})} \tag{65}$$

From Equation (60)

$$t(x^+, 1) - t_{mm} = \frac{11}{24} \, r_0 q / k \tag{66}$$

which when substituted into Equation (65) gives

$$Nu = \frac{48}{11} \cong 4.36 \tag{67}$$

Substitution of (64), (58) and (53) into (65) yields the local value of the Nusselt Modulus for the case $q(x^+)$ = constant.

$$Nu = \frac{1}{\frac{11}{48} + \frac{1}{2}\sum_m \frac{e^{-\gamma_m^2 x^+}}{\gamma_m^4 \, H'(-\gamma_m^2)}} \tag{68}$$

CALCULATION FOR LINEARLY VARYING WALL TEMPERATURES

In similar fashion the use of the boundary condition $T_w(x^*) - T_o = A\,x^*$ where A = any constant, gives:

$$q(x^+) = \frac{Ak}{4r_o} + \frac{2Ak}{r_o} \sum_m \frac{C_m}{2} \frac{R_m'(1)}{\lambda_m^2} e^{-\lambda_m^2 x^+} \tag{69}$$

$$T_{mm}(x^+) - T_o = A x^+ - \frac{88}{768} A - 8A \sum_m \frac{C_m}{2} \frac{R_m'(1)}{\lambda_m^4} e^{-\lambda_m^2 x^+} \tag{70}$$

$$Nu = \frac{\frac{1}{2} + 4\sum_m \frac{C_m}{2} \frac{R_m'(1)}{\lambda_m^2} e^{-\lambda_m^2 x^+}}{\frac{88}{768} + 8\sum_m \frac{C_m}{2} \frac{R_m'(1)}{\lambda_m^4} e^{-\lambda_m^2 x^+}} \tag{71}$$

APPROXIMATIONS FOR SMALL x^+

Whenever x^+ is small, a large number of the terms in the series, Equation (3) must be taken. The Leveque solution is a good approximation for such cases. As shown by Tribus and Klein,[2] the wall temperature and heat flux for such a case are related by

$$q(x) = \frac{k P_r^{1/3}}{3\Gamma(4/3)} \left(\frac{\rho}{9\mu}\right)^{1/3} \left(\frac{du}{dy}\right)_{y=0}^{1/3} \int_0^x (x-\xi)^{-1/3} dt_w(\xi) \tag{72}$$

and

$$t_w(x) - t_o = \frac{2P_r^{-1/3}}{3k\Gamma(2/3)} \left(\frac{\rho}{9\mu}\right)^{-1/3} \left(\frac{du}{dy}\right)_{y=0}^{-1/3} \int_0^x \frac{q(\xi)\,d\xi}{(x-\xi)^{2/3}} \tag{73}$$

For flat ducts

$$\left(\frac{du}{dy}\right)_{y=0} = 3u_m/b \tag{74}$$

for round ducts

$$\left(\frac{du}{dy}\right)_{y=0} = 4u_m/r_o \tag{75}$$

Substitution of Equation (75) into Equations (72) or (73) (and noting that the mixed mean temperature of the fluid is essentially equal to its inlet value at small values of x^*) gives for the three cases under consideration:

For constant wall temperature:

$$Nu = \frac{2 \cdot 2^{1/3}\, x^{+\,-1/3}}{9^{1/3}\,\Gamma(4/3)} = 1.3565\, x^{+\,-1/3} \qquad x^{+} \leq 0.001 \tag{76}$$

For constant heat flux

$$Nu = \frac{2^{1/3}\, 9^{1/3}\, \Gamma(5/3)}{3}\, x^{+\,-1/3} = 1.6393\, x^{+\,-1/3} \tag{77}$$

For linearly varying wall temperature

$$Nu = \frac{3 \cdot 2^{1/3}\, x^{+\,-1/3}}{9^{1/3}\,\Gamma(4/3)} = 2.0348\, x^{+\,-1/3} \tag{78}$$

Figure 1 shows a graph of the functions R_0, R_1, and R_2 compared with solutions given by Jakob. Figure 2 shows the variations in Nusselt modulus for three cases

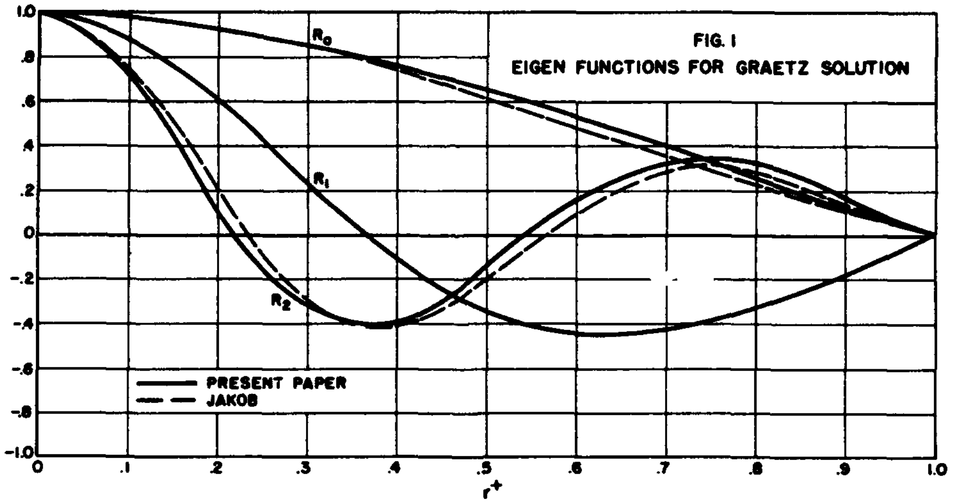

FIG. 1
EIGEN FUNCTIONS FOR GRAETZ SOLUTION

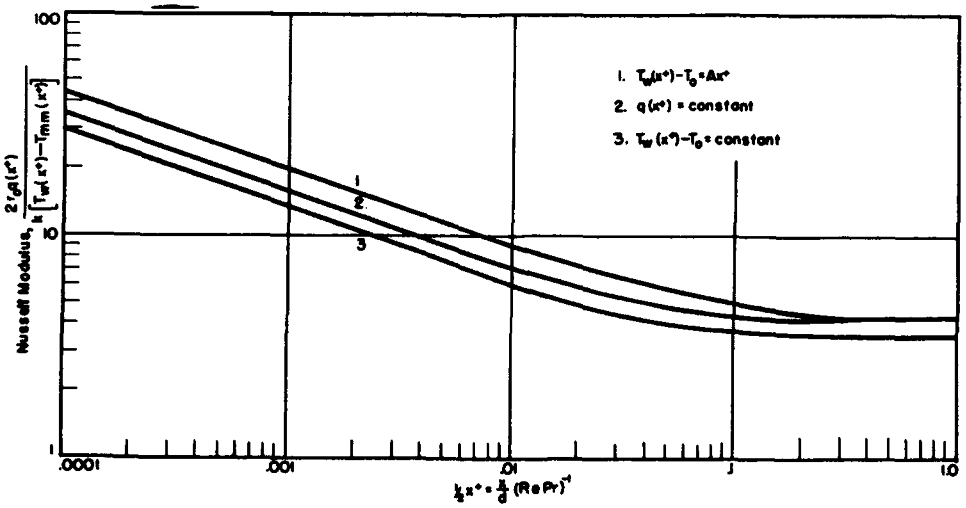

Fig.2. Laminar Flow of a Constant Property Fluid in a Round Tube

3 'Uber die Warmeleitungsfahigkeit von Flussigkeiten,' by L. Graetz, Annalen der Physik Chem., vol. 25, 1885, pp. 337-357.

(1) wall temperature constant,

(2) heat flux constant, and

(3) wall temperature increasing linearly along the pipe wall.

The Nusselt modulus is defined by the equation

$$Nu = \frac{q(x)}{t_w(x) - t_{mm}'} \frac{2r_o}{k} \tag{79}$$

The mixed-mean temperature, t_{mm}, is determined by integrating the heat flux from the origin ($X^+ = 0$) to the position where q(x) is known.

CONCLUSIONS

The methods used in this paper have a wide applicabilit r example, the liquid metals systems analyzed by Poppendiek[4] could be tr d by the methods used here. The unsymmetrical boundary conditions treated by Yih and Cermak[5] can also be readily treated by these methods.

The authors are somewhat surprised at the fact that whereas the asymptotic formulae are all supposed to be valid only for very large λ, in actuality values of n as small as 4 seem to give excellent results. The reasons for the good results are not now clear.

APPENDIX A

The equations for a flat duct system with walls at $y = \pm b$ (A-1)

Defining $Re = 4U_m \rho b/\mu$, $x^+ = (x/b)(RePr)^{-1}$, $y^+ = (y/b)$ the equation to be solved is

$$\frac{3}{8} \frac{\partial t}{\partial x^+} = \frac{1}{1-y^{+2}} \frac{\partial^2 t}{\partial y^{+2}} \tag{A-2}$$

which has a solution

$$\theta = \sum_{m=0}^{\infty} K_m Y_m(y^+) e^{-8/3 \lambda_m^2 x^+} \tag{A-3}$$

satisfying $\Theta = 1$ at $x^+ = 0$, $\theta \to 0$, $x^+ \to 0$, if $y(y^+)$ satisfies

[4] 'Forced Convection Heat Transfer in Thermal Entrance Regions, Part 1,' by H. F. Poppendiek, Oak Ridge National Laboratory, Tenn., ORNL-913, Physics, series A, March, 1951.

[5] 'Laminar Heat Convection in Pipes and Ducts,' by C. S. Yih and J. E. Cermak, Civil Engineering Department, Colorado Agricultural and Mechanical College, Fort Collins, Colo., September, 1951. ONR Contract No. N90 nr 82401, NR063-071/1-19-49.

$$Y'' + \lambda^2 (1 - y^{+2}) Y = 0 \tag{A-4}$$

with $Y'(0) = Y(1) = 0$, $Y(0) = 1$. λ_n is the value of λ to permit $Y_n(1) = 0$. The coefficients K_n are given by

$$K_m = \frac{-2}{\lambda_m \left(\frac{\partial Y_m}{\partial \lambda}\right)_{y^+=1,\ \lambda=\lambda_m}} \tag{A-5}$$

By the methods in the text the WKB approximation is found to be

$$Y(y^+) = \frac{\cos\left\{\lambda \int_0^{y^+} (1 - s^2)^{1/2}\, ds\right\}}{1 - y^{+2}} \tag{A-14}$$

for $0 \leq y^+ < 1$.

Defining $z = 1 - y^+$, the solution of A(A-4) for $z \ll 1$ is found to be

$$Y(z) = \tfrac{1}{2} (\lambda \pi z)^{1/2} \left\{ \sin\left(\frac{\pi\lambda}{4} - \frac{\pi}{12}\right) J_{1/3}\left(\frac{\lambda\sqrt{8}}{3} z^{3/2}\right) - \sin\left(\frac{\lambda\pi}{4} - \frac{5\pi}{12}\right) J_{-1/3}\left(\frac{\lambda\sqrt{8}}{3} z^{3/2}\right)\right.$$

The eigenvalues are

$$\lambda_m = 4m + 5/3 \tag{A-25}$$

$$\left(\frac{\partial Y_m}{\partial \lambda}\right)_{\substack{\lambda=\lambda_m \\ y^+=1}} = (-1)^{m+1} \frac{\pi^{1/2} \lambda_m^{1/6}}{3^{2/3}\, \Gamma(2/3)\, 2^{1/6}} \tag{A-29}$$

$$K_m = (-1)^m \frac{3^{2/3}\, \Gamma(2/3)\, 2^{13/6}}{\pi^{1/2}} \lambda_m^{-7/6} \tag{A-30}$$

$$\left(\frac{d Y_m}{d y^+}\right)_{y^+=1} = (-1)^{m+1} \frac{\pi^{1/2}\, 2^{1/6}\, \lambda_m^{5/6}}{3^{5/6}\, \Gamma(4/3)} \tag{A-31}$$

$$q(x^+) = -\frac{k}{b} (t_w - t_0) \sum_m K_m Y_m'(1)\, e^{-\frac{8}{3}\lambda_m^2 x^+} \tag{A-32}$$

To obtain the fluid temperature for a given heat flux use

$$t - t_0 = \frac{b}{k} \int_{s=0}^{x^+} \bar{g}(x^+ - s, y^+)\, q(s)\, ds \tag{A-44}$$

The integrating kernel, $\bar{g}$, is given by

$$g(x^+, y^+) = \frac{2}{3} - \sum_m \frac{e^{-\gamma_m^2 x^+}}{\gamma_m^2\, \bar{H}'(-\gamma_m^2)} - \sum_m K_m Y_m(y^+) \sum_m \frac{e^{-\gamma_m^2 x^+}}{\frac{8}{3}\lambda_m^2 - \gamma_m^2} \tag{A-48}$$

where the $-\gamma^2_m$ are the zeroes of

$$\bar{H}(s) = -\sum_m \frac{K_m Y_m'(1)}{s + \frac{8}{3}\lambda_m^2} \tag{A-47}$$